Angular Momentum Of Rigid Body

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1 Introduction

Given a particle a with momentum p_a, angular momentum of a can be defined as the linear momentum about some radius s_a.

$$L_a = s_a \times p_a \tag{1}$$

With many particles composing a rigid body, we can define angular momentum of the body as the total angular momentum of all particles in the body.

$$L_{tot} = \sum_i s_i \times p_i \tag{2}$$

While linear momentum is concisely written in Euclidean space, angular momentum of a rigid body is simply put in cylindrical space. That is, if we spin about the $\hat{z}$-axis the position of a particle can be written in cylindrical coordinates as:

$$\vec{s} = r\hat{r} + z\hat{z}$$

The velocity of a particle:

$$\frac{d\vec{s}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z} + z\dot{\hat{z}}$$

Since our spinning body is rigid, we know $\dot{r} = \dot{z} = 0$. Also, since $\hat{\theta} \times \hat{z} = 1$, it is always orthogonal to $\hat{r}$, making $\hat{z}$ independent of t. Therefore, $\dot{\hat{z}} = 0$ too.

$$\frac{d\vec{s}}{dt} = r\dot{\theta}\hat{\theta}$$

Given our cylindrical expressions of position and velocity, we can substitute them for position and velocity in the second equation.

$$L_{tot} = \sum_i (r_i\hat{r} + z_i\hat{z}) \times m_i(r_i\dot{\theta}\hat{\theta}) \tag{3}$$

$$L_{tot} = \sum_i [\hat{z}(r_i^2\dot{\theta}m_i) - \hat{r}(z_i r_i\dot{\theta})]$$

Only interested in the angular momentum about the $\hat{z}$-axis, we take the dot product:

$$L_z = L_{tot} \cdot \hat{z} = \sum_i m_i r_i^2 \dot{\theta}_z \tag{4}$$

If we let $I = \sum_i m_i r_i^2$, then $L_z = I\dot{\theta}_z$. In more familiar notation, $L_z = I\omega_z$. Angular momentum becomes a powerful tool once combined with torque, which we can found by taking the first derivative of angular momentum with respect to time.

$$\tau_z = \frac{dL_z}{dt} = I\alpha_z$$

Much how a force can be understood in terms of cause and effect (such as $ma = F = mg = -kd$), torque on spinning bodies have the same feature. τ_z can represent any external torque on our body along the $\hat{z}$-axis, which will always be related to $r \times F = \tau_z = I\alpha_z$.